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SOLUTION OF THE HYPERBOLIC HEAT-CONDUCTION EQUATION BY EXPANSION IN A SMALL PARAMETER

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A method of finding the solution of the hyperbolic heat-conduction equation as a power series of a small parameter (the relaxation time) is discussed.

In the hyperbolic heat-conduction equation [1]

$$\frac{\partial T}{\partial \tau} + \tau_r \frac{\partial^2 T}{\partial \tau^2} = a \frac{\partial^2 T}{\partial x^2} \tag{1}$$

the relaxation time τ_r is small. For example in aluminum $\tau_r = 10^{-11}$ sec. Hence, one can consider (1) as an equation of a small parameter $\varepsilon = \tau_r/\tau_0$ and use asymptotic methods for its analysis and solution [2, 3].

We consider (1) (written in dimensionless form) for the following initial and boundary conditions:

$$T(x,\ 0) = \theta_0(x),\ \frac{\partial T(x,\ 0)}{\partial \tau} = \theta_1(x), \tag{2}$$

$$\beta_{i1} \frac{\partial T((i-1)l,\ \tau)}{\partial x} + (-1)^i \beta_{i2} T((i-1)l,\ \tau) = \varphi_i(\tau),\ i = 1,\ 2, \tag{3}$$

where depending on the type of boundary condition, the constants β_{i1}, β_{i2} are either equal to zero or correspond to the appropriate thermal constants.

Because (1) has the small parameter ε as a coefficient of the higher-order derivative, a power series expansion of the solution in ε must contain boundary-layer type terms depend-

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ing on the variable $\eta = \tau/\varepsilon$ [4]. We look for an approximate solution $T_\varepsilon(x, \tau)$ in the form

$$T_\varepsilon(x, \tau) = T_0(x, \tau) + \varepsilon T_1(x, \tau) + \varepsilon^2 T_2(x, \tau) + \ldots + \Pi_0(x, \eta) + \varepsilon \Pi_1(x, \eta) + \ldots \tag{4}$$

Substituting (4) into (1)-(3) and equating the coefficients of identical powers of ε, we obtain the following family of boundary-value problems:

$$\frac{\partial T_0}{\partial \tau} = \mathrm{Fo}\frac{\partial^2 T_0}{\partial x^2}, \tag{5}$$

$$T_0(x, 0) = \theta_0(x) - \pi_0(x),\ \Gamma[T_0] = \varphi(\tau),\ \varphi(\tau) = (\varphi_1(\tau), \varphi_2(\tau));$$

$$\frac{\partial T_k}{\partial \tau} = \mathrm{Fo}\frac{\partial^2 T_k}{\partial x^2} - \frac{\partial^2 T_{k-1}}{\partial \tau^2},\ k = 1, 2, \ldots, \tag{6}$$

$$T_k(x, 0) = 0,\ \Gamma[T_k] = 0;$$

$$\frac{\partial \Pi_0}{\partial \eta} + \frac{\partial^2 \Pi_0}{\partial \eta^2} = 0,$$

$$\Pi_0(x, 0) = \pi_0(x),\ \frac{\partial \Pi_0(x, 0)}{\partial \eta} = 0; \tag{7}$$

$$\frac{\partial \Pi_k}{\partial \eta} + \frac{\partial^2 \Pi_k}{\partial \eta^2} = \mathrm{Fo}\frac{\partial^2 \Pi_{k-1}}{\partial x^2},\ k = 1, 2, \ldots,$$

$$\Pi_k(x, 0) = 0,\ \frac{\partial \Pi_k(x, 0)}{\partial \eta} = h_k(x), \tag{8}$$

$$h_1(x) = \theta_1(x) - \frac{\partial T_0(x, 0)}{\partial \tau},\ h_k(x) = -\frac{\partial T_{k-1}(x, 0)}{\partial \tau},\ k \geqslant 2. \tag{9}$$

where Γ is the operator corresponding to the boundary conditions (3). The function $\pi_0(x)$ is an arbitrary smooth function such that $\Gamma[\pi_0] = 0$. Thus, the coefficients in expansion (4) can be found if we can solve the boundary-value problems (5)-(8) subject to condition (9).

We assume that $\varphi_1(\tau)$, $\varphi_2(\tau)$ are differentiable the required number of times. The solution of the boundary-value problem (5) is written in the form

$$T_0(x, \tau) = a(x)\varphi_1(\tau) + b(x)\varphi_2(\tau) + u(x, \tau),$$

where $a(x)$ and $b(x)$ are smooth functions (linear for boundary conditions of types I and III) which satisfy the relations

$$\beta_{11}a'(0) - \beta_{12}a(0) = 1,\ \beta_{11}b'(0) - \beta_{12}b(0) = 0,$$

$$\beta_{21}a'(l) + \beta_{22}a(l) = 0,\ \beta_{21}b'(l) + \beta_{22}b(l) = 1.$$

The function $u(x, \tau)$ satisfies the following equation with homogeneous boundary conditions:

$$\frac{\partial u}{\partial \tau} = \mathrm{Fo}\frac{\partial^2 u}{\partial x^2} + F(x, \tau),$$

$$u(x, 0) = u_0(x), \tag{10}$$

$$\Gamma[u] = 0,$$

$$F(x, \tau) = \mathrm{Fo}(a''(x)\varphi_1(\tau) + b''(x)\varphi_2(\tau)) - a(x)\varphi_1'(\tau) - b(x)\varphi_2'(\tau),$$

$$u_0(x) = \theta_0(x) - \pi_0(x) - a(x)\varphi_1(0) - b(x)\varphi_2(0).$$

The solution of (10) can be written in the form [5]

$$u(x, \tau) = \sum_{n=1}^{\infty}\left\{u_n \exp(-\lambda_n^2\tau) + \int_0^{\tau} f_n(s)\exp[-\lambda_n^2(\tau-s)]\,ds\right\} X_n(x), \tag{11}$$

where $\{\lambda_n^2\}$ and $\{X_n(x)\}$ are the eigenvalues and eigenfunctions of the differential operator $\partial^2/\partial x^2$ with homogeneous boundary conditions, and $\{u_n\}$ and $\{f_n(\tau)\}$ are the Fourier coefficients of the functions $u_0(x)$ and $F(x, \tau)$:

$$u_n = \int_0^l u_0(x) X_n(x)\,dx, \quad f_n(\tau) = \int_0^l F(x, \tau) X_n(x)\,dx.$$

We assume that the initial conditions of (5) are such that one can calculate the required derivatives of $T_0(x, \tau)$ by termwise differentiation of the series (11). Substituting the expression for $\partial^2 T_0/\partial\tau^2$ into (6) for k = 1, we obtain a boundary-value problem for $T_1 \times (x, \tau)$. This problem is of the form (10) in which $u_0(x) = 0$, $F(x, \tau) = \partial^2 T_0/\partial\tau^2$. Hence, $T_1(x, \tau)$ can be calculated using (11). The higher-order functions $T_2(x, \tau)$, $T_3(x, \tau)$, ... are calculated recursively with the help of (11).

We consider in detail a type I boundary-value problem for the heat-conduction equation (1). We take the case of homogeneous boundary conditions

$$T(0, \tau) = T(l, \tau) = 0,$$

and we further put $\pi_0(x) \equiv 0$. The solution of (5) in this case is [6]

$$T_0(x, \tau) = \sum_{n=1}^{\infty} c_n \exp\left[-\left(\frac{\pi n}{l}\right)^2 \mathrm{Fo}\,\tau\right] \sin\frac{\pi n}{l}x,$$
$$c_n = \frac{2}{l}\int_0^l \theta_0(\xi)\sin\frac{\pi n}{l}\xi\,d\xi. \tag{12}$$

Differentiating (12) twice with respect to τ and substituting the result into (6) for k = 1, we find, solving the inhomogeneous equation

$$T_1(x, \tau) = -\tau\sum_{n=1}^{\infty} c_n\left(\frac{\pi n}{l}\right)^4 \mathrm{Fo}^2 \exp\left[-\left(\frac{\pi n}{l}\right)^2 \mathrm{Fo}\,\tau\right]\sin\frac{\pi n}{l}x = -\tau\frac{\partial^2 T_0}{\partial\tau^2}. \tag{13}$$

Similarly it is straightforward to find $T_2(x, \tau)$, $T_3(x, \tau)$:

$$T_2(x, \tau) = 2\tau\frac{\partial^3 T_0}{\partial\tau^3} + \frac{\tau^2}{2}\frac{\partial^4 T_0}{\partial\tau^4},$$
$$T_3(x, \tau) = -5\tau\frac{\partial^4 T_0}{\partial\tau^4} - 2\tau^2\frac{\partial^5 T_0}{\partial\tau^5} - \frac{\tau^3}{6}\frac{\partial^6 T_0}{\partial\tau^6}. \tag{14}$$

The form of (13), (14) shows that functions $T_k(x, \tau)$ for $k \geqslant 1$ can be sought in the form

$$T_k = \Phi_k T_0, \quad \Phi_k = (-1)^k\sum_{i=1}^{k} A_i^{(k)}\tau^i\frac{\partial^{k+i}}{\partial\tau^{k+i}}, \tag{15}$$

where the $A_i^{(k)}$ are found by the method of undetermined multipliers after substitution of (15) into (6).

Recursion relations for the $A_i^{(k+1)}$ for some values of i are:

$$A_1^{(k+1)} = 2A_1^{(k)}, \quad A_2^{(k+1)} = \frac{1}{2}A_1^{(k)} + A_2^{(k)} + 3A_3^{(k)}, \quad \ldots, \quad A_{k+1}^{(k+1)} = \frac{1}{k+1}A_k^{(k)}.$$

The above method of solving (6) based on (15) is useful not only for type I boundary conditions but also for general boundary conditions given by the operator Γ when the functions on the right-hand side (φ_1 and φ_2,) are independent of τ.

We now determine the coefficients of the singular part of the expansion (4). Solving the boundary-value problem (7), we obtain

$$\Pi_0(x) = \pi_0(x) \equiv 0. \tag{16}$$

With the help of the method of variation of parameters, the solution of the boundary-value problem (8) is written as

$$\Pi_k(x, \eta) = [1 - \exp(-\eta)]\, h_k(x) + \mathrm{Fo} \int_0^{\eta} [1 - \exp(s - \eta)] \frac{\partial^2 \Pi_{k-1}(x, s)}{\partial x^2}\, ds. \tag{17}$$

From (17) it is straightforward to obtain

$$\Pi_1(x, \eta) = [1 - \exp(-\eta)]\, h_1(x), \quad \Pi_2(x, \eta) = [1 - \exp(-\eta)]\, h_2(x) + \mathrm{Fo}\, [\eta - 2 + (\eta + 2)\exp(-\eta)]\, h_1''(x), \tag{18}$$

$$\Pi_3(x, \eta) = [1 - \exp(-\eta)]\, h_3(x) + \mathrm{Fo}\, [\eta - 2 + (\eta + 2)\exp(-\eta)]\, h_2''(x) +$$
$$+ \mathrm{Fo}^2 \left[\frac{\eta^2}{2} - 3\eta + 6 - \left(\frac{\eta^2}{2} + 3\eta + 6\right)\exp(-\eta)\right] h_1^{(\mathrm{IV})}(x),$$

which shows that $\Pi_k(n, \eta)$ can be sought in the form

$$\Pi_k(x, \eta) = [1 - \exp(-\eta)]\, h_k(x) + \mathrm{Fo}\, [\eta - 2 + (\eta + 2)\exp(-\eta)]\, h_{k-1}''(x) +$$
$$+ \mathrm{Fo}^2 \left[\frac{\eta^2}{2} - 3\eta + 6 - \left(\frac{\eta^2}{2} + 3\eta + 6\right)\exp(-\eta)\right] h_{k-2}^{(\mathrm{IV})} + \ldots \tag{19}$$
$$\ldots + \mathrm{Fo}^{k-1} h_1^{(2k-2)}(x)\, [B_{k-1}^{(k)} \eta^{k-1} + \ldots + B_0^{(k)} + (C_{k-1}^{(k)} \eta^{k-1} + \ldots + C_0^{(k)}) \exp(-\eta)],$$

and the coefficients $B_{k-1}^{(k)}, \ldots, B_0^{(k)}, C_{k-1}^{(k)}, \ldots, C_0^{(k)}$ are determined recursively in terms of the corresponding coefficients with superscript $(k-1)$ using the method of undetermined multipliers after substitution of (19) into (17).

Thus, for the case of boundary conditions given by the operator Γ with right-hand sides independent of τ, the functions $T_k(x, \tau)$ and $\Pi_k(x, \tau)$, $k = 1, 2, \ldots$, are determined in terms of derivatives of $T_0(x, \tau)$. The series (4) together with (15), (16), and (19) represents a formal expansion of the solution of the hyperbolic heat-conduction equation. Questions on the convergence of this series require further study, but (4) with a finite number of terms can serve as a good approximation to the exact solution of the hyperbolic equation.

As an example, we consider the transfer of heat in a semiinfinite medium when the heat propagation speed is finite [1].

Equation (1) ($\tau_r = \varepsilon$) is to be solved subject to the boundary conditions

$$T(x, 0) = \partial T(x, 0)/\partial \tau = 0, \quad T(0, \tau) = 1. \tag{20}$$

The solution of (5) for this case is

$$T_0(x, \tau) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/(2\sqrt{a\tau})} \exp(-s^2)\, ds.$$

Calculating $\partial^2 T_0/\partial \tau^2$ we find the function $T_1(x, \tau)$ from (15). We than have the first approximation $T_{\varepsilon 1}(x, \tau)$ to the solution of the hyperbolic heat-conduction equation (1) with boundary conditions (20):

$$T_{\varepsilon 1}(x, \tau) = T_0(x, \tau) + \frac{x\varepsilon}{2\sqrt{\pi a}\, \tau^{3/2}} \left(\frac{3}{2} - \frac{x^2}{4a\tau}\right) \exp\left(-\frac{x^2}{4a\tau}\right).$$

Calculation of the derivatives of $\partial^3 T_0/\partial \tau^3$ and $\partial^4 T_0/\partial \tau^4$ allows us to obtain $T_2(x, \tau)$ and the second approximation $T_{\varepsilon 2}(x, \tau)$:

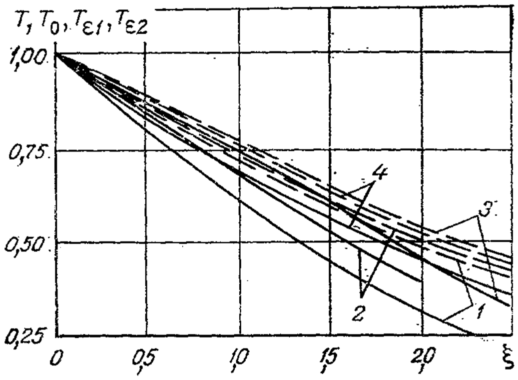

Fig. 1. Comparison of the ε dependence of the solution T_0 of the parabolic heat-conduction equation (curve 1), the exact solution of the hyperbolic heat-conduction equation (curve 2), the approximations $T_{\varepsilon 1}$ (curve 3) and $T_{\varepsilon 2}$ (curve 4) for different values of τ. The solid curves refer to τ = 2ε, the dashed curves to τ = 5ε.

$$T_{\varepsilon 2}(x,\ \tau) = T_0(x,\ \tau) + \frac{x\varepsilon}{2\sqrt{\pi a}\,\tau^{3/2}}\left(\frac{3}{2} - \frac{x^2}{4a\tau}\right)\exp\left(-\frac{x^2}{4a\tau}\right) +$$

$$+ \frac{x\varepsilon^2}{2\sqrt{\pi a}\,\tau^{5/2}}\left[\frac{1}{2}\left(\frac{x^2}{4a\tau}\right)^3 - 2\left(\frac{x^2}{4a\tau}\right)^2 + 5\,\frac{x^2}{4a\tau} - \frac{15}{8}\right]\exp\left(-\frac{x^2}{4a\tau}\right).$$

We compare the approximations $T_{\varepsilon 1}(x, \tau)$ and $T_{\varepsilon 2}(x, \tau)$ with the exact solution of problem (1), (20) given in [1]. Putting $\xi = x/\sqrt{a\varepsilon}$, we note that for $\xi > \tau/\varepsilon$ we have $T(x, \tau) = 0$. The calculations show that for $\tau/\varepsilon < 1$ the functions $T_{\varepsilon 1}$ and $T_{\varepsilon 2}$ poorly approximate the exact solution the exact solution T. For $\tau/\varepsilon = 1$, $\xi \leqslant 1$ the function $T_{\varepsilon 2}$ more closely approximates T, while the function $T_{\varepsilon 1}$ less closely approximates T in comparison with T_0. For $\tau/\varepsilon \geqslant 10$ differences in the functions T, T_0, T_{ε_1}, T_{ε_2} are small for all values of ξ. In Fig. 1, the dependence of T, T_0, $T_{\varepsilon 1}$, $T_{\varepsilon 2}$ on ξ is shown for $\tau/\varepsilon = 2$ and $\tau/\varepsilon = 5$.

Hence analysis of the approximation $T_{\varepsilon 1}$ and $T_{\varepsilon 2}$ show that in the calculation of the temperature field in a semiinfinite medium, for $\tau \geqslant \varepsilon$, $\xi < \tau/\varepsilon$ the formulas obtained by the small parameter method result in a good approximation to the exact solution of the hyperbolic heat-conduction equation.

NOTATION

T, temperature; τ, time; x, coordinate; τ_0, time scale; a, thermal diffuxivity; l, layer thickness of the body; $\mathrm{Fo} = a\tau_0/l^2$, Fourier number; $T_{\varepsilon N} = \sum_{i=0}^{N}(\varepsilon^i T_i + \varepsilon^i \Pi_i)$; superscript k, approximation number in the small parameter power series expansion.

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